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THE PREDICTION OF TURBULENT PRANDTL AND SCHMIDT NUMBERS

A. J. REYNOLDS
Department of Mechanical Engineering, Brunel University, Uxbridge, Middlesex, England

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Abstract—This paper examines more than thirty ways of predicting the relationship between turbulent transfers of momentum and a passive contaminant such as heat or dissolved matter. The models are divided into seven classes, on the basis of method of derivation or field of application. Three classes comprise modifications of the simplest mixing-length model to allow for diffusion during the lateral motion of the fluid element which is conceived to carry the transferred entities. The other four classes are more heterogeneous: formal analyses based on Reynolds equations; results derived from various expressions for the eddy diffusivities; several kinds of model applicable to wall layers in particular; and purely empirical formulae representing limited data.

An attempt has been made to assess the utility and potential for development of these models, both from the practical point of view, that of devising accurate heat- and mass-transfer formulae, and from the fundamental point of view, that of gaining an understanding of the actual transferring mechanisms. There is a plethora of formulae that can, with suitable choices of empirical constants, represent the gross features of experimental data. However, only the formal analyses account in a consistent way for the pressure interactions which influence momentum transfer. At present, formal results are available only for the degenerate case of weak, decaying turbulence, although they do prove useful in suggesting the limiting behaviour to be expected in more general flows.

NOMENCLATURE

a, radius of spherical element of fluid;
A^+, empirical constant for wall layer;
b, empirical constant;
B^+, empirical constant for wall-layer temperature field;
c_f, friction coefficient;
c_p, constant-pressure thermal capacity;
c_v, constant-volume thermal capacity;
C, C_1 to C_4, empirical constants with differing values in various formulae;
d, pipe diameter;
D, molecular diffusivity of matter;
e, fluctuation in specific internal energy;
h, fluctuation in specific enthalpy;
I, measure of intensity of turbulence;
K, von Kármán's constant;
K', constant with role of K for temperature field;
K_h, effective diffusivity of heat;
K_m, effective diffusivity of momentum;
K_p, molecular diffusivity of a general property P;
l, mixing length or distance travelled by fluid element;
l_h, damped mixing length for heat transfer;
l_m, damped mixing length for momentum transfer;
L, length scale for roller;
L_E, Eulerian integral length scale;
L_L, Lagrangian integral length scale;
m, empirical value of exponent;
n, empirical or predicted value of exponent, or integer in summations;
N_R, Reynolds number characterizing larger scales of turbulence;
Nu, Nusselt number;
p, pressure fluctuation;
P, historical value of general property for moving element;
$\bar{P}$, (time-) mean variation of general property for surroundings of moving element;
Pe, Péclet number;
Pr, molecular Prandtl number;
Pr_e, effective value of Prandtl number;
Pr_t, turbulent Prandtl number;
R, radius of pipe;
Re, Reynolds number;
R_E, Eulerian correlation coefficient;
R_L, Lagrangian correlation coefficient;
R_{uv}, correlation coefficient between velocity fluctuations;
$R_{v\theta}$, correlation coefficient between velocity and temperature fluctuations;
s, strain parameter;
Sc, molecular Schmidt number;
Sc_t, turbulent Schmidt number;
t, time;
T, (time-) mean temperature;
T_E, Eulerian integral time scale;
T_L, Lagrangian integral time scale;
T^+, mean temperature scaled using wall-layer parameters;
u, velocity fluctuation in direction of mean flow;
u_f, friction velocity, $(\tau_w/\rho)^{\frac{1}{2}}$;
U, (time-) mean velocity or characteristic velocity of roller;
U_a, mean or bulk velocity for pipe flow;
U_1, free-stream velocity at edge of boundary layer;

U^+, mean velocity scaled using friction velocity, U/u_f;
v, velocity fluctuation, typically normal to mean flow and in direction of transfer;
v', RMS intensity of velocity fluctuation or other characteristic value;
V_e, effective convection velocity for energy;
V_e', typical value of V_e;
y, coordinate measured from and normal to wall;
y_f, wall-layer length scale, ν/u_f;
y^+, scaled coordinate, y/y_f.

Greek symbols

α_0, diffusivity representing small-scale turbulent transfer of heat;
δ, boundary-layer thickness or spatial separation between correlated points;
δ_h, parameter characterizing transfer of heat from a moving element;
δ_m, parameter characterizing transfer of momentum from a moving element;
δ_p, parameter characterizing transfer of a general property P from a moving element;
ε_e, eddy diffusivity characteristic of gradient diffusion of energy;
ε_h, eddy diffusivity of heat;
ε_m, eddy diffusivity of momentum: eddy viscosity;
ε_n, eddy diffusivity for transfer normal to wall;
ε_t, eddy diffusivity for transfer tangential to wall;
ε_D, eddy diffusivity of matter;
θ, temperature fluctuation;
θ', typical value of temperature fluctuation for turbulent convection;
κ, molecular diffusivity of heat;
λ, microscale characterizing velocity fluctuations;
λ_θ, microscale characterizing temperature fluctuations;
ν, molecular diffusivity of momentum: kinematic viscosity;
ν_0, diffusivity representing small-scale turbulent transfer of momentum;
ρ, fluid density, taken to be essentially constant;
τ, time in correlation calculations;
τ_w, wall shear stress;
ψ, distortion factor accounting for effect of "entity" shape on momentum transfer;
ψ', distortion factor for heat transfer.

1. INTRODUCTION

IT IS a tribute to man's ingenuity and individuality that more than thirty analytical models of turbulent transport processes have been devised within the last twenty-five years. It is the purpose of this paper to review this work, both to assist engineers faced with transfer calculations, and to provide future investigators with an understanding of current achievements. Studies of this problem are so numerous and so widely dispersed that one can hardly hope to have found every relevant example, but those that have been located should serve to indicate the main lines of development.

Most analyses begin with calculations of the eddy diffusivities of momentum and heat or passively convected matter. From these can be calculated the turbulent Prandtl or Schmidt numbers, the ratios of the eddy diffusivity of momentum—the eddy viscosity—to those of enthalpy and mass:

$$Pr_t = \varepsilon_m/\varepsilon_h \quad \text{and} \quad Sc_t = \varepsilon_m/\varepsilon_D. \tag{1.1}$$

These ratios parallel exactly the molecular Prandtl and Schmidt numbers:

$$Pr = \nu/\kappa \quad \text{and} \quad Sc = \nu/D. \tag{1.2}$$

We shall fix our attention on the predictions of the eddy-diffusivity ratios (1.1) provided by the several models of turbulent transfer processes. In using these ratios as measures of performance, we are in one way asking little of an analytical model, and in another, demanding a great deal. We ask little in that the spatial variations of the individual diffusivities need not be predicted accurately. We demand a good deal by concentrating on the model's ability to distinguish between the transfers of momentum and of a convected property, that is, to predict departures of Pr_t and Sc_t from unity.

In fact, many of the Prandtl number formulae to be considered do involve a ratio of eddy and molecular diffusivities (ε_m/ν or ε_h/κ). This is not a severe limitation, since the distribution of eddy viscosity is known with adequate precision for some often-studied flows; an example is the semi-empirical prediction of Travis *et al.* [1] for the case of fully developed pipe flow. Thus many analysts do not attempt to predict the eddy-viscosity variation within the flow, but adopt a semi-empirical expression for it at a convenient point in the analysis.

We shall restrict attention to temperature and concentration ranges that are sufficiently small to ensure that there is no significant interaction between the flow and the transfer of heat or mass within it; in particular, we shall neglect both buoyancy and the compressibility effects arising in high-speed gas flows. Accordingly, we assume an almost perfect analogy between heat and mass transfers, with the relationships $Pr_t = f(Pr)$ and $Sc_t = f(Sc)$ having the same functional form. For gases this assumption is suspect, for the turbulent enthalpy flux which is usually taken as a measure of heat flux is

$$\overline{hv} = \overline{ev} + \overline{pv}/\rho \tag{1.3}$$

where h and e are the enthalpy and internal-energy fluctuations, and v is the velocity fluctuation in the direction considered. Here the energy flux is separated into terms representing the passive convection of internal energy and a work interaction involving pressure and velocity. It is not plausible to assume that e and p are correlated with v in the same way. But when this assumption is made, the pressure-work term contributes the fraction $(c_p - c_v)/c_p$ to the enthalpy flux (c_p and c_v are specific heats). The difference between

actual and assumed enthalpy fluxes may be expected to be of the same order, negligible for liquids, but around one-third for gases. Even in the latter case, the difference between the transfer processes for enthalpy and for passively conveyed matter is hardly large enough to be detected within the usual experimental variability.

1.1. *Experimental evidence*

Detailed comparisons between experiment and analysis and between alternative analytical models will not be attempted in this paper. Most of the formulae to be considered incorporate constants, or even functions, that have been selected to make each formula conform with a limited body of data, and the difficulties of measurement are such that we are still comparing (sometimes, contrasting) results obtained by different experimenters.

Some existing reviews of measured ratios Pr_t and Sc_t are: for wall flows, Blom [2], Simpson *et al.* [3], Monin and Yaglom [4], and Eckert and Drake [5]; for core flows, Groenhoff [6] and Huetz [7]; and for free turbulence, Reynolds [8]. Without doubt there are differences in the ratios typical of these three classes of flow. Moreover, in each case there is evidence that the ratio depends on the corresponding molecular value, Pr or Sc, on position within the flow, and on the local turbulence intensity. The last two factors are sometimes represented by the single quantity ε_m/ν and sometimes by separate parameters, such as y/R and $Re_d = U_a d/\nu$. (Here pipe flow has been considered for definiteness, $d = 2R$ being the diameter and U_a the bulk velocity.)

We can summarize these observations in a formula for Pr_t which displays the trends most often found in measurements:

$$Pr_t = C_1 \exp[-C_2 Pr^m (\varepsilon_m/\nu)^n] \tag{1.4}$$

with C_1, C_2, m and n positive constants. This proposal is consistent with an increase in Pr_t as the wall is approached and ε_m/ν falls; this trend seems, on balance, to be the more likely. Greater generality could be obtained by providing an explicit dependence on position, through a parameter such as y/R or y/y_f ($y_f = \nu/u_f$ is the wall-layer length scale, u_f being the friction velocity). The postulate (1.4) also demonstrates the widely observed, roughly reciprocal relationship between the molecular and turbulent diffusivity ratios:

$$Pr_t \gtrless 1 \quad \text{for} \quad Pr \lessgtr 1 \quad \text{unless} \quad Pr \sim 1. \tag{1.5}$$

Within the wall layer there are large differences ($\varepsilon_t/\varepsilon_n \sim 10$) between the eddy diffusivities for transfers tangential to and normal to the wall. Here we shall concern ourselves only with the normal diffusivities; readers interested in tangential transfer, or in transfers within channels of complex cross-section, may refer to Quarmby and Quirk [9, 10] and Ramm and Johannsen [11].

1.2. *Classification of models*

The most numerous group of analytical results are three classes based on Prandtl's concept of the mixing length:

(a) Jenkins' [12] analysis and subsequent modifications;

(b) Deissler's [13] analysis and developments of it; and

(c) more varied models, generally combining features of the preceding and adopting a more detailed picture of the mixing.

The remaining results are more heterogeneous; they range from simple empiricism to detailed statistical calculations. We can distinguish four groups:

(d) formal analyses based on Reynolds momentum equations and allied relationships among correlation functions for the fluctuating quantities;

(e) diffusivity models starting from formal expressions for the eddy diffusivity based on gradient-diffusion arguments or their generalization;

(f) wall-layer models applicable to that particular flow; and

(g) empirical formulae describing a limited body of data.

Class (f) differs from the others in being defined by a field of application. Some of its models use "renewal-penetration" arguments for the sublayer, while others use mixing lengths "damped" in the manner of van Driest.

2. MIXING-LENGTH MODELS WITH DIFFUSION

2.1. *General character*

These analyses seek to account for the exchange of momentum and heat (or convected matter) between the instantaneous surroundings and a lump of fluid which moves across gradients of velocity and temperature (or concentration). At the end of its journey the lump—almost always taken to be a sphere in detailed calculations—is usually assumed to mix instantaneously with its new surroundings; the transfer is calculated using the terminal properties of the moving element. Clearly, this model is inconsistent, in a way that the original mixing-length argument is not, in that no account is taken of that fraction of the transferred property which is conveyed only part way along the mixing length. However, since empirical constants are usually introduced at the end of the analysis, this inconsistency is of little practical importance.

The assumption that $Pr_t = 1$ is most radically in error for liquid–metal heat transfer, as a consequence of the very large molecular thermal diffusivities of liquid metals, which give $Pr \ll 1$. Some workers interested in predicting heat transfer in these fluids have taken account of the transfer of heat from the laterally-moving fluid element, while neglecting the roughly analogous effect which small-scale diffusion has on momentum transfer. Accordingly, their results give a poor account of fluids with large and moderate values of Pr and Sc.

The simplest way of treating the entire range of molecular conductivities is to adopt identical expressions for the two eddy diffusivities; thus

$$Pr_t = f(\varepsilon_m/\kappa)/f(\varepsilon_m/\nu) \text{ or } Pr\, g(\varepsilon_m/\kappa)/g(\varepsilon_m/\nu) \tag{2.1}$$

the structure of the functions f and g depending on the details of the model. These results indicate that

$$Pr_t \to 1 \quad \text{as} \quad Pr \to 1 \tag{2.1}$$

a prediction only very crudely in accord with experiment. This unrealistic behaviour follows from the use of identical transfer laws—typically, simple molecular diffusion—for momentum and heat transfers from the moving element, no account being taken of the pressure interactions which must influence the momentum transfer. For practical purposes this problem can be resolved by introducing a constant multiplier into equations (2.1). However, this expedient serves only to mask a fundamental failing of models of this kind.

Table 1. Jenkins-type mixing-length models

Authors	Functional form	Empirical constants	Validity
Jenkins [12]	$f(Pr, \varepsilon_m/\nu)$	0	All Pr
Sleicher and Tribus [15]	$f(Pr, \varepsilon_m/\nu)$	$f(\varepsilon_m/\nu)$	All Pr
Rohsenow and Cohen [17]	$f(Pr)$	1	$Pr \ll 1$
Tien [16]	$f(Pr, \varepsilon_m/\nu)$	1	All Pr
Sénéchal [18]	$f(a^2v'/l\kappa)$	1	$Pr < 1$

2.2. *Jenkins-type models: Table 1*

Jenkins [12] considered the motion of a spherical element of fluid across uniform velocity and temperature gradients, taking the radius a to be equal to the mixing length l and the time of flight to be l/v', with v' the lateral fluctuation in velocity. He adopted an existing molecular-diffusion formula for a solid sphere to find the mean temperature on arrival and used a parallel result to predict the momentum transfer from the sphere. There are several ambiguities in Jenkins' work, notably the use of an apparently incorrect form of the basic diffusion formula. Using the proper result (Carslaw and Jaeger [14], p. 235), we find that Jenkins' analysis leads to

$$Pr_t = \frac{1}{Pr} \frac{1-(90/\pi^2)\sum_{n=1}^{\infty} n^{-4}\exp(-n^2\pi^2 lv/a^2v')}{1-(90/\pi^2)\sum_{n=1}^{\infty} n^{-4}\exp(-n^2\pi^2 l\kappa/a^2v')}. \tag{2.2}$$

(The restriction to $l = a$ has not yet been made.) The general character of the prediction is not altered by this modification to the solution.

While Jenkins' argument leaves no parameters that can be adjusted to match experiments, the other models listed in Table 1 are less rigid. Sleicher and Tribus [15] noted that Jenkins' predictions were generally too high, and simply factored his formula:

$$Pr_t(Pr) = \frac{Pr_t(\text{air, experimental})}{Pr_t(\text{air, Jenkins})} \times Pr_t(Pr, \text{Jenkins}). \tag{2.3}$$

Tien [16] revised Jenkins' model in another way, arguing that a smaller sphere radius should be used; he suggested $l/a = 2$, a value since adopted by other workers. Tien also pointed out that the neglect of turbulent transfers to and from the moving sphere implies that

$$Pr(v'\lambda_\theta/\nu) = Pr^{1/2}(v'\lambda/\nu) \ll 1 \tag{2.4}$$

where λ and λ_θ are the lengths (microscales) characterizing the smaller elements of the velocity and temperature fluctuations. Tien showed that these conditions could not be satisfied in air ($Pr \sim 1$) and were only marginally satisfied in liquid metals ($Pr \ll 1$).

Rohsenow and Cohen [17] modified Jenkins' result in a more fundamental way. They analysed test results to show that the effect of ε_m/ν was small (actually, Pr_t varied over the range 2 to 4 at least, with Pr essentially constant), and then sought a formula dependent only on the molecular Prandtl number. Thus

$$Pr_t^{-1} = 27{\cdot}8Pr\left[1-(90/\pi^2)\sum_{n=1}^{\infty} \times n^{-4}\exp(-Cn^2\pi^2/Pr)\right] \tag{2.5}$$

with $C = 0{\cdot}0024$.

The model used by Mme. Sénéchal [18], as described by Huetz [7], is basically like Jenkins', but specifies the boundary conditions on the moving sphere in a different way, and leaves the ratio a/l unspecified. As in Rohsenow and Cohen's analysis, the role of the molecular viscosity is rejected, the turbulent Prandtl number being found to depend on a single parameter.

2.3. *Deissler-type models: Table 2*

In Deissler's first attack on this problem [13], attention is again fixed on a sphere moving normal to the mean flow. Being interested in very small molecular Prandtl numbers, he took account only of heat transfer from the moving element. The model is somewhat less explicit than Jenkins' but, like it, takes the sphere radius to be proportional to the mixing length, and assumes molecular diffusion. With the heat-transfer coefficient assumed constant, integration of the equation expressing the energy balance for the sphere leads to

$$Pr_t^{-1} = bPe[1-\exp(-1/bPe)] \tag{2.6}$$

with Pe a mean-flow Péclet number, assumed proportional to that for the moving element. For pipe flow, $Pe = U_a d/\kappa$, and the empirical constant is assigned the

Table 2. Deissler-type mixing-length models

Authors	Functional form	Empirical constants	Validity
Deissler [13]	$f(Pe)$	1	$Pr \ll 1$
Lykoudis and Touloukian [19]	$f(Pr)$	1	$Pr \ll 1$
Aoki [20]	$f(Re^{2\cdot25}Pr)$	1	$Pr \ll 1$
Mizushina and Sasano [21]	$f(Pr, \varepsilon_m/\nu)$	3	$Pr < 1$
Mizushina *et al.* [23]	$f(Pr, \varepsilon_m/\nu, a/l)$	3	$Pr < 1$
Wassel and Catton [24]	$f(Pr, \varepsilon_m/\nu)$	3	All Pr

Table 3. More varied mixing-length models

Authors	Functional form	Empirical constants	Validity
Azer and Chao [25,26]	$f(Re, Pr, y/R)$	1+	$\begin{cases}0\cdot6 < Pr < 50\\ Pr \ll 1\end{cases}$
Buleev [28]	$f(Re, Pr, \text{etc.})$	5	All Pr
Dwyer [27]	$f(Pr, \varepsilon_m/\nu)$	2	$Pr \ll 1$
Tyldesley and Silver [30]	$f(Pr)$	1	All Pr
Tyldesley [31]	$f(Pr, ?)$	2	All Pr
Ramm and Johannsen [11]	$f(Re, Pr, \text{etc.})$	5+	All Pr

value $b = 0\cdot000153$. In this analysis the diffusion is assumed to take place through a layer of constant thickness, contrary to the nature of simple diffusive processes; it is this assumption that introduces the exponential function of equation (2.6) and the modifications of it to be considered below.

Deissler's model differs fundamentally from Jenkins' in taking the transfers to and from the sphere to be controlled by diffusive processes external to the moving element, rather than within it. As we shall see, this step allows the model-constructor greater freedom in specifying the transfer mechanisms and, in particular, gives him the ability to prescribe momentum transfer by a law different from that used for heat transfer.

Lykoudis and Touloukian [19] modified Deissler's analysis in several respects, in particular, by choosing the transfer time in such a way that the length scale of the motion did not influence the small-scale transfers. Their detailed result is

$$Pr_t^{-1} = (6/\pi^2) \sum_{n=1}^{\infty} n^{-2} \exp(-n^2 C/Pr) \qquad (2.7)$$

with $C = 0\cdot01$.

Aoki [20] followed Deissler's analysis almost to the end, but then introduced a transfer law intended to account for small-scale turbulent mixing, obtaining

$$Pr_t^{-1} = KRe^{0\cdot45}Pr^{0\cdot2}[1-\exp\{-1/(KRe^{0\cdot45}Pr^{0\cdot2})\}] \qquad (2.8)$$

with the constant $K = 0\cdot014$ for pipe flow.

Mizushina and Sasano [21] also followed Deissler's pattern, save that Ranz and Marshall's [22] heat-transfer law for an isolated sphere in a uniform stream was introduced:

$$Nu = 2 + 0\cdot6Re^{1/2}Pr^{1/3}. \qquad (2.9)$$

The assumption that $\varepsilon_m/\nu \propto Re$ (for the sphere) led to

$$Pr_t^{-1} = (lv'/\varepsilon_m)\phi[1-\exp(-1/\phi)] \qquad (2.10)$$

with

$$\phi = C_1(\varepsilon_m/\nu)Pr/\{1 + C_2(\varepsilon_m/\nu)^{1/2}Pr^{1/3}\}.$$

The value $lv'/\varepsilon_m = 1\cdot5$ was adopted, and C_1 and C_2 were chosen to match experimental results.

Mizushina *et al.* [23] returned to this problem, now keeping the ratio a/l explicit in their formulae. They specified it in different ways in the core (a/l constant) and in the wall layer [$a/l \propto (\varepsilon_m/\nu)^{-1/3}$], and thus obtained different formulae for Pr_t in these two parts of the flow.

The step of introducing an analogous variation for ε_m/ν was taken by Wassel and Catton [24]. They also retained the more widely interpretable ε_m/ν instead of the Péclet number of equation (2.6), and introduced a few more adjustable constants:

$$Pr_t^{-1} = (C_1 Pr/C_3)\frac{1-\exp\{-C_2/(Pr\,\varepsilon_m/\nu)\}}{1-\exp\{-C_4/(\varepsilon_m/\nu)\}}. \qquad (2.11)$$

The constants C_1 to C_4 were assigned the values 0·21, 5·25, 0·20 and 5 in order to make the formula match experimental results for wall layers in air.

2.4. *More varied mixing-length models: Table* 3

The analytical results to be considered now were developed in the knowledge of some of the contributions discussed above; typically, they adopt a more sophisticated, or at least more detailed picture of the turbulent activity. Some of these allow a variability in the distances and directions travelled by the moving fluid elements, and do not explicitly assume instantaneous mixing with the terminal surroundings. Thus the defining feature of this category is, not the use of an immutable mixing length, but the introduction of a discrete element which gains or loses heat and momentum as it moves through the body of the fluid.

In the work of Azer and Chao [25], experimental

values for pipe flow were introduced for various characteristics of the turbulence, and a more detailed treatment was given to several aspects of the mixing process. However, the analysis is constrained by the assumption of a basic analogy between momentum and heat transfers, expressed in the mixing-length result

$$Pr_t = \frac{(u_2-u_1')/(u_2-u_1)}{(T_2-T_1')/(T_2-T_1)} = \frac{1+(T_1'-T_1)/(T_2-T_1')}{1+(u_1'-u_1)/(u_2-u_1')} \quad (2.12)$$

where the subscripts 1 and 2 denote time-mean values at the initial and final positions, and the prime denotes the arrival values for the moving element, as modified by in-flight diffusion. Although their complete formulae are complicated, Azer and Chao gave simpler approximations for pipe flow:

$$Pr_t = \frac{1+57f(y/R)/(Re^{0\cdot46}Pr^{0\cdot58})}{1+135f(y/R)/Re^{0\cdot45}} \quad (2.13)$$

valid for $Pr = 0{\cdot}6$ to 15, and

$$Pr_t = \frac{1+380f(y/R)/Pe^{0\cdot58}}{1+135f(y/R)/Re^{0\cdot45}} \quad (2.13)$$

valid for liquid metals. In both cases, the variation across the pipe is given by

$$f(y/R) = \exp\{-(y/R)^{1/4}\}.$$

Azer [26] applied the same basic model to annulus flow.

More limited aims were set by Dwyer [27]. He sought only to find an average value of the turbulent Prandtl number, in order to modify Lyon's well-known heat-transfer law for liquid metals in pipes:

$$Nu = 7+0{\cdot}025(Pe/Pr_t)^{0\cdot8} \quad (2.14)$$

in which it had earlier been assumed that $Pr_t = 1$. Dwyer assumed that the surface area of the moving fluid element was proportional to some power of ε_m/ν, that conduction to the element took place through a film of constant thickness, proportional to the mixing length, and that Stokes' law gave the drag of the moving element. Thus he obtained

$$Pr_t^{-1} = 1-1{\cdot}82Pr^{-1}(\varepsilon_m/\nu)^{-1\cdot4} \quad (2.15)$$

where ε_m/ν is to be given its maximum value within the flow, and the constants have been chosen to make equation (2.14) agree with heat-transfer measurements for pipe flow. This formula has the odd feature of becoming negative for small values of Pr and ε_m/ν. Although the formula (2.15) is that usually ascribed to him, Dwyer went on to develop some alternative predictions for Pr_t, using transfer laws from the moving sphere with forms more appropriate to small-scale turbulence.

The analysis of Buleev [28] involved an even more detailed specification of the small-scale transfers. He sought to account for the three-dimensional character of the turbulence, so that the results might be applied in more general channel sections, not merely in round pipes. This intricate analysis is difficult to summarize, but the essential features are: establishing and integrating linear equations for the velocity and temperature variations of a moving element; forming the transfer correlations using these solutions; integrating over space to find the net transfers; and finally determining the eddy diffusivities using these integrals.

Ramm and Johannsen [11] developed Buleev's model further, finding it necessary to introduce a turbulence length scale dependent on direction, in order to account for the large differences between normal and tangential diffusivities.

Buleev mentions some earlier work by Voskresenskii and Turilina [29], implying that it is somewhat similar to Deissler's [13] analysis.

Tyldesley and Silver [30] departed from the pattern set by earlier workers by considering a more general moving "entity"; in practice, it was taken to be an ellipsoid. They introduced distortion factors into the drag and transfer equations in order to allow for non-spherical forms, and also showed that the selection of a solid rather than a fluid element had a rather small effect on the results. Their procedures are, broadly speaking, those adopted by Buleev, although his treatment is more general in some respects.

Although some aspects of their model are relatively sophisticated, the laws adopted by Tyldesley and Silver to represent transfers between their entities and the surroundings are the simplest molecular-diffusion results: Stokes' law and $Nu = 2$, the latter strictly applicable to a sphere in still fluid [compare equation (2.9)]. Their prediction is

$$Pr_t = 2/3+2(\psi/\psi')/9Pr \quad (2.16)$$

with ψ and ψ' distortion factors accounting for the effect of element shape on transfers of momentum and heat. In many of their results, Tyldesley and Silver set $\psi'/\psi = 1$, appropriate to a sphere, but not too much in error for any "entity" that is not very elongated; this value is adopted in the results quoted below.

To account for the possibility that the transfer from the moving element be accomplished by turbulence, Tyldesley and Silver applied the result (2.16) to the element itself:

$$(Pr_t)_2 = 2/3+2/9Pr_t = (2+9Pr)/(3+9Pr). \quad (2.17)$$

Of course, if this operation is justified once, it can be justified again and again as the turbulence intensity and range of length scales increase. The results tend to $(Pr_t)_n = 0{\cdot}9$ for large n, for all values of Pr.

The formulae (2.16, 17) describe only the limiting cases of very weak and very intense turbulence, although these restricted results have the happy feature of being virtually independent of empirical inputs. Tyldesley [31] allowed for arbitrary intensity by replacing Pr_t on the right of equations (2.17) by the effective value

$$Pr_e = (\varepsilon_m+\nu)/(\varepsilon_h+\kappa). \quad (2.18)$$

This gave

$$Pr_t = (2/3)\frac{I+1+1/3Pr+(3/2)IPr/(1+3Pr)}{I+1} \quad (2.19)$$

where $I = (2/81)\psi N_R^2$ is a measure of intensity, N_R being a Reynolds number characteristic of the larger scales

of the turbulence. The result (2.19) has as its limits the formulae (2.16, 17) when $I \to 0$ and ∞.

A second generalization of Tyldesley [31] was the introduction of differing distortion factors for the three co-ordinate directions. Since there is evidence of elongated structures in free turbulence, Tyldesley introduced values for the ψ's that are appropriate to a prolate spheroid whose length is some ten times its diameter, in order to obtain results applicable to free-turbulent flows. The resulting formula is rather like equation (2.19), but the modification to the coefficients leads to values of Pr_t that are smaller by a factor of 0·75 to 0·85. Thus $Pr_t = 0{\cdot}6$ to 0·9 is predicted for free turbulence, in rough agreement with experiment. Tyldesley went on to predict even lower values of Pr_t (0·55 to 0·6) for highly intermittent free turbulence. However, Reynolds [8] has found that higher values (1·0 to 1·5) are found in reality, and this aspect of Tyldesley's work does not seem to be successful.

3. ALTERNATIVES TO THE MIXING-LENGTH MODEL

The procedures to be discussed now are so varied that little can be said of them as a group, save that they differ from what have been called mixing-length models in not starting with a discussion of transfers to and from a distinct fluid element moving across gradients of velocity and temperature or concentration. However, the first two classes have in common with most mixing-length analyses the assumption that the mean-property gradients are uniform over distances comparable with significant lateral displacements of more-or-less coherent fluid masses. This assumption is ill-founded in most cases of practical interest, but it is possible that the introduction of empirical values will compensate for this deficiency.

3.1. *Formal analyses based on exact equations: Table* 4

The difficulties of a formal analysis of turbulent shear flow are so formidable that the only convincing products of rigorous mathematical analysis relate to isotropic turbulence, just a few tentative steps having been taken towards cases of wider practical importance. Thus Corrsin [33] briefly examined transfers in shear flow, following a more detailed study of heat transfer in isotropic turbulence. He utilized the isotropic result

$$\lambda_\theta = (2/Pr)^{1/2}\lambda \tag{3.1}$$

linking the microscales used earlier in equations (2.4), and obtained

$$Pr_t = (1/5)(R_{uv}/R_{v\theta})^2 \tag{3.2}$$

where R denotes a correlation coefficient. This result he supposed to be independent of the molecular Pr, but most of the evidence that has come to light since that time fails to justify this assumption.

Dunn and Reid [34] discussed isotropic turbulence in the final period of decay, where the Reynolds and Péclet numbers are very small, and triple correlations become negligible so that an exact analysis is possible. According to Deissler [35], their result for a linear gradient of mean temperature gives

$$Pr_t^{-1} = 2{\cdot}05\frac{Pr}{1-Pr}\left\{1-\left(\frac{2Pr}{1+Pr}\right)^{3/2}\right\}. \tag{3.3}$$

The limiting values from this result are

$$Pr_t = 0{\cdot}27,\ 0{\cdot}65 \text{ and } 0{\cdot}49/Pr \text{ for } Pr \to \infty,\ 1 \text{ and } 0 \tag{3.4}$$

in accord with the empirical conclusions (1.5).

Table 4. Formal analyses based on Reynolds equations

Authors	Functional form	Restrictions
Corrsin [33]	f(correlations)	Isotropy; large Re and Pe
Dunn and Reid [34]	$f(Pr)$	Isotropy; small Re and Pe
Deissler [35]	$f(Pr, t\,dU/dy)$	Homogeneity; small Re
Tyldesley [32]	$f(Pr, ?)$	Converted to mixing-length model
Hill [37]	—	Direct-interaction approximation

In a final contribution Tyldesley [32] began by applying Fourier transformations to the equations of motion, and then developed his arguments with reference to wave-number space. However, his results were interpreted in such a way that they became equivalent to those obtained earlier from the "entity" model, and it is appropriate to consider them here. In this new interpretation, the result (2.19) is replaced by

$$Pr_t = (2/3)\frac{\nu_0/\nu + 1 + 1/3Pr + (1/3)(\nu_0/\nu)/(\nu_0/\alpha_0)}{\nu_0/\nu + 1}. \tag{2.20}$$

Here ν_0 and α_0 are diffusivities representing small-scale turbulent transfers between an entity and its surroundings. While Tyldesley held that the results (2.19, 20) should be exactly equivalent, there is nothing in the development that demands that this be the case.

The intensity parameter $I = \nu_0/\nu$ is presumably related to measurable quantities such as ε_m/ν and Re_d, but no explicit connection has been suggested. This is the significance of the functional form $f(Pr, ?)$ given opposite Tyldesley's work in Table 3. An extended account of Tyldesley and Silver's work has been given since it has some interesting features, but the failure to define the measure of intensity does reduce the practical utility of the results. Lawn and Walker [30, 31] have made a number of comments on this method of analysis.

In extending Dunn and Reid's work to include a linear variation of mean velocity, Deissler [35] also assumed triple correlation functions to be negligible. Consequently, the turbulence still decays, though its behaviour is now dependent on the dimensionless strain parameter $s = t\,dU/dy$, with t the time during which the initially isotropic turbulence has been subjected to straining. In the presence of a mean velocity gradient, the truncation of the full equations by rejection of higher-order interactions is a very suspect procedure.

Deissler's predictions for the diffusivities are obtained by integrating wave-number spectra over the wave-number range, and no explicit formula for Pr_t is given. The general character of the results is indicated in Table 5, where the values for $s = 0$ are obtained from equation (3.3). Although these results tend to a uniform

Table 5. Deissler's predictions of Pr_t for strained homogeneous turbulence

$s = t\,dU/dy$	Pr		
	10	1	0·01
0	0·304	0·650	49·8
1	0·272	0·542	—
2	0·240	0·425	15·2
5	0·292	0·435	5·65
10	0·450	0·600	3·75
20	0·578	0·750	3·31
50	0·666	0·908	3·54

value (unity) when the Reynolds number of strain parameter takes on a large value, the values calculated for $Pr > 1$ at first display the opposite tendency. To put this behaviour into perspective, we may make use of Deissler's attempt to relate his strain parameter to conditions in pipe flows. He estimated that $s = t\,dU/dy \sim 0{\cdot}05U_a d/\nu$, whence $Re_d \sim 200$ for $s = 10$. Thus it appears that the anomalous behaviour is confined to a strain régime which is not relevant to maintained wall turbulence, for which $Re_d > 2000$. No such clear-cut conclusion can be drawn for free turbulence, since free shear layers become turbulent at much lower Reynolds numbers than do wall-bounded flows.

The problem of transfers in isotropic turbulence has also been considered by Kokorev [36].

Hill [37] has investigated the possibility of applying the direct-interaction approximation, developed by Kraichnan and others, to the calculation of eddy diffusivities. While it is not possible to give a formal justification of the direct-interaction hypotheses, comparisons for simpler problems have shown some of its predictions to be more realistic than those given by other schemes of approximation, in particular, the "series method" used by Deissler [35]. Such comparisons are made by Leslie [38]. While the computing requirements for direct-interaction calculations are very great, there is some hope of obtaining results for simple cases such as those dealt with by Dunn and Reid and by Deissler; the former case would provide a check on the direct-interaction techniques. Hill has given a progress report on his attempts to carry out calculations of this kind; great difficulties were encountered with the numerical procedures.

3.2. *Diffusivity models: Table 6*

These analyses start from formal expressions for the eddy diffusivities which incorporate, sometimes tacitly, the assumption that the turbulence is homogeneous over distances within which the mean properties change significantly or the assumption that the mean gradients are uniform for distances over which the fluctuations are significantly correlated.

In seeking to express the diffusivity of momentum in terms of measurable quantities, Tien [39] started from the results

$$\varepsilon_m = \overline{v^2}\int_0^\infty R_L(\tau)\,d\tau = v'L_L \tag{3.5}$$

where $\overline{v^2} = v'^2$ is the intensity of lateral fluctuations, and L_L and R_L are the Lagrangian lateral length scale and velocity correlation coefficient. Since he did not see any reason to distinguish between the values of L_L for the several transfer processes, Tien concluded that $Pr_t = 1$.

In their first attack on this problem, Bobkov *et al.* [40] started in much the same way as Tien, writing the thermal diffusivity as

$$\varepsilon_h = \overline{v^2}\int_0^\infty R_L(\tau)\,d\tau = \overline{v^2}T_L \tag{3.6}$$

where T_L is the Lagrangian integral time scale. They related this to measurable Eulerian values through the postulates

$$R_L(\tau) = R_E(\tau,\delta) = R_E(\delta,0)R_E(0,\tau)$$
$$= \exp(-\tau/T_E)\exp(-\delta/L_E)$$

Table 6. Diffusivity models

Authors	Validity	Nature of model
Tien [39]	$Pr \sim 1$	Diffusivity integral without in-flight diffusion
Bobkov *et al.* [40]	$Pr \ll 1$	Diffusivity integral without in-flight diffusion
Bobkov and Ibragimov [41]	$Pr < 10$	Empirical extension
Simpson *et al.* [3]	—	Allowance for bulk convection by large "eddies"
Reynolds [43]	All Pr	Diffusivity integral with in-flight diffusion

where T_E and L_E are Eulerian integral scales. When the spatial separation was expressed as $\delta = 0{\cdot}8v'\tau$, evaluation of the integral in equations (3.6) gave

$$\varepsilon_h = \overline{v^2} T_L = \frac{\overline{v^2} T_E}{1 + 0{\cdot}8v' T_E / L_E}. \tag{3.7}$$

In developing their arguments further, Bobkov and Ibragimov [41] noted that the diffusivity might better be expressed as

$$\varepsilon_h = \overline{v^2} \int_0^\infty R_L(\tau) f(\kappa, \tau)\, \mathrm{d}\tau \tag{3.8}$$

but returned to their earlier postulate (3.6) in the absence of information regarding the transfer function f. Subbotin *et al.* [42] extended this line of investigation to channels of complex cross-section, and presented a number of relevant experimental results.

The developments of Reynolds [43] were intended to elucidate the effects of the molecular diffusivities and the overall level of turbulent activity, rather than to determine the way in which Pr_t or Sc_t varied within a particular flow. The starting point was a particular form of equation (3.8), namely

$$\varepsilon_p = \overline{v^2} \int_0^\infty R_L(\tau) \exp(-\delta_p \tau)\, \mathrm{d}\tau. \tag{3.9}$$

This was derived, in the manner of Schubauer and Tchen [44], from a postulated equation for the transfer of the property P between the moving element and its historical surroundings:

$$\mathrm{d}P/\mathrm{d}t = -\delta_p (P - \bar{P}) \tag{3.10}$$

subject to the requirement that the mean gradient $\mathrm{d}\bar{P}/\mathrm{d}y$ changes only slowly with y. The assumption that $R_L(\tau) = \exp(-\tau/T_L)$ led to

$$\varepsilon_p = \frac{\overline{v^2} T_L}{1 + \delta_p T_L}$$

whence

$$Pr_t = \frac{1 + \delta_h T_L}{1 + \delta_m T_L}. \tag{3.11}$$

The time constant for the transfer process, $1/\delta_p$, can be related to the defining parameters by applying equation (3.10) to a "roller" which conveys the property P across the gradient $\mathrm{d}\bar{P}/\mathrm{d}y$. Taking the transfer from the roller to be accomplished by molecular diffusion, and taking $T_L \sim L/U$ to relate the time scale and the length and velocity scales of the roller, Reynolds obtained

$$T_L \delta_p \propto (Re\, \nu/K_p)^{-1/2} \tag{3.12}$$

where K_p is the molecular diffusivity of P. There follows

$$Pr_t = \frac{1 + C_1/Pe^{1/2}}{1 + C_2/Re^{1/2}}. \tag{3.13}$$

The values $C_1 = 86$ and $C_2 = 200$ are suggested as representative of the core region in pipe flow.

This analysis can be generalized in the manner of some of the mixing-length models. To provide an indication of variations across the flow, we can argue that $UL/\nu \propto \varepsilon_m/\nu$ rather than $\propto Re$, and so obtain

$$Pr_t = \frac{1 + C_1/(Pr\, \varepsilon_m/\nu)^{1/2}}{1 + C_2/(\varepsilon_m/\nu)^{1/2}}. \tag{3.14}$$

To represent more varied transfer processes between the roller and its environment, we might introduce effective diffusivities:

$$\begin{aligned} K_m &= \nu(1 + C_3 Re^{1/2}) \\ K_h &= \kappa(1 + C_4 Re^{1/2} Pr^{1/3}). \end{aligned} \tag{3.15}$$

The particular forms are suggested by the drag and heat-transfer laws for a sphere.

The formulae (3.13, 14) give $Pr_t \to 1$ as Re or $\varepsilon_m/\nu \to \infty$. This unrealistic behaviour can be avoided by introducing a non-diffusive (on the small scale) contribution into the momentum transfer; thus

$$\varepsilon_m = \frac{\overline{v^2} T_L}{1 + \delta_m T_L} - C_3 \overline{v^2} T_L. \tag{3.16}$$

Taking $C_3 = 0$ for the thermal diffusivity, we obtain

$$Pr_t = (1 + C_1 Pe^{-1/2}) \left\{ \frac{1}{1 + C_2 Re^{-1/2}} - C_3 \right\}. \tag{3.17}$$

The values $C_1 = 100$, $C_2 = 120$ and $C_3 = 0{\cdot}15$ are representative of the core of pipe flow, and give $Pr_t \to 0{\cdot}85$ as $Re \to \infty$. As would be expected, the constants C_1 and C_2 are more nearly equal in this formulation.

Another way of distinguishing between the mechanisms of momentum and heat transfer was used by Simpson *et al.* [3] in analysing the outer part of a boundary layer. They followed Townsend's [45] suggestion that heat transfer be represented as the sum of gradient-diffusion and bulk-convection terms (the latter representing the activity of the larger "eddies"), while momentum transfer be given by simple gradient diffusion. Thus the correlation between velocity and temperature fluctuations is

$$-\overline{v\theta} = \varepsilon_h \mathrm{d}T/\mathrm{d}y = \varepsilon_e \mathrm{d}T/\mathrm{d}y + \overline{V_e \theta} \tag{3.18}$$

with ε_e representing gradient diffusion of energy, and V_e representing large-eddy convection.

Simpson *et al.* supposed that $\varepsilon_e = \varepsilon_m$ and took the convective element to be the product of typical fluctuations:

$$\overline{V_e \theta} \sim V_e' \theta'. \tag{3.19}$$

The effective convection velocity was supposed to vary linearly through the boundary layer: $V_e' \sim (V_e')_1 y/\delta$ with δ the thickness of the layer to $U = 0{\cdot}99U_1$. The temperature fluctuation was represented as

$$\theta' \sim l\, \mathrm{d}T/\mathrm{d}y \tag{3.19}$$

with l the mixing length. These postulates led to

$$Pr_t = \frac{\varepsilon_m}{\varepsilon_h} = \frac{\varepsilon_m}{\varepsilon_m + (V_e')_1 l y/\delta}. \tag{3.20}$$

This result is independent of Pr, and $Pr_t < 1$, for the boundary layer, at least. The results obtained, for air and for $0{\cdot}1 < y/\delta < 1$, were nearly identical to those

given by Rotta's [46] empirical formula, which will be considered below.

Note that the modifications of the diffusivity expressed by equations (3.16, 18) are fundamentally different. The former provides a non-diffusive transfer within the turbulence, the latter a non-diffusive transfer by the turbulence. However, the effect of the restrictions (3.19) is to restore the proposal (3.18) to the basic gradient-diffusion form, with $\overline{v\theta} \propto \mathrm{d}T/\mathrm{d}y$, contrary to the original concept. While these two modifications have the effect of nudging the diffusivity ratio into better agreement with experiment, neither gives a convincing account of the role of pressure interactions or of turbulent convection which cannot be represented as gradient diffusion.

length representation of the wall layer. His approach has been extended to the temperature distribution near a wall by Cebeci [51], who took the eddy diffusivity to be given by

$$\varepsilon_h = -\overline{v\theta}/(\mathrm{d}T/\mathrm{d}y) = l_m l_h \mathrm{d}U/\mathrm{d}y$$

with

$$l_m = K[1-\exp(-y^+/A^+)]$$

and

$$l_h = K'[1-\exp(-y^+(\sqrt{Pr})/B^+)]$$

The turbulent Prandtl number can now be calculated as

$$Pr_t = l_m/l_h = (K/K')\frac{1-\exp(-y^+/A^+)}{1-\exp[-y^+(\sqrt{Pr})/B^+]}. \qquad (3.23)$$

Table 7. Wall-layer models

Authors	Functional form	Validity
Marchello and Toor [7]		All Pr
Taccoen [54]	$f(Pr, y^+)$	$Pr \ll 1$
Thomas [48, 50]	$f(Pr, \varepsilon_m/\nu, y^+)$	Not $Pr \gg 1$
Blom [2]	$f(U^+)$	All Pr
Cebeci [51, 53]	$f(Pr, y^+)$	$Pr \sim 1$
Na and Habib [52]	$f(Pr, y^+)$	All Pr

3.3. *Wall-layer models: Table 7*

One subclassification which can be distinguished is that of models that apply Higbie's concept of diffusive penetration through a film, in this case the viscous sublayer, intermittently renewed by fluid from the region of turbulent flow. Marchello and Toor [47] were not very successful in their analysis along these lines; they obtained results which differ from most other predictions in giving $Pr_t \gtrless 1$ for $Pr \gtrless 1$, contrary to the experimental trends (1.5).

Thomas [48] made a more successful application of the renewal-penetration model, basing his analysis on earlier work by Danckwerts and Hanratty. He took the near-wall temperature gradient to be

$$\mathrm{d}T^+/\mathrm{d}y^+ = Pr\exp\{-(Pr\tfrac{1}{2}c_f)^{1/2}y^+\} \qquad (3.21)$$

corresponding to the velocity gradient

$$\mathrm{d}U^+/\mathrm{d}y^+ = \exp\{-(\tfrac{1}{2}c_f)^{1/2}y^+\}. \qquad (3.21)$$

The superscript + indicates that the usual wall-layer scaling has been carried out ($y^+ = y/y_f$, etc.), and $c_f = \tau_w/(\frac{1}{2}\rho U_1^2)$ is the friction coefficient at the wall. The effective Prandtl number is given by

$$Pr_e = (\mathrm{d}T^+/\mathrm{d}y^+)/(\mathrm{d}U^+/\mathrm{d}y^+)$$

and the relationship (2.18) between Pr_e and Pr_t leads to

$$Pr_t = \frac{Pr+\nu/\varepsilon_m}{1+\nu/\varepsilon_m}\cdot\frac{\exp\{-(Pr\tfrac{1}{2}c_f)^{1/2}y^+\}}{\exp\{-(\tfrac{1}{2}c_f)^{1/2}y^+\}}. \qquad (3.22)$$

This line of argument was extended to a developing boundary layer by Thomas and Fan 3[49] and to liquid metals by Thomas [50].

Another distinct group of wall-layer analyses are those developed from van Driest's damped mixing-

In developing this model further, Na and Habib [52] set $K' = K$, and treated B^+ as a function of Pr, in order to deal with fluids other than air. Cebeci [53] showed their function $B^+(Pr)$ graphically, and also indicated how the various constants (K, K', A^+ and B^+) depend upon Reynolds number.

There remain a few studies of the wall region that do not fall into the classes considered above. Taccoen [54] has described the wall layer by dividing it into discontinuous segments, in the manner of von Kármán. Thus a highly discontinuous function $Pr_t = f(y^+, Pr)$ is defined by the variations $U^+ = f(y^+)$ and $T^+ = f(Pr\, y^+)$. Another wall-layer result is reported by Wassel and Catton [24] are attributed to Blom:

$$Pr_t = 1 - \frac{x^4/24}{e^x - 1 - x - x^2/2 - x^3/6} \qquad (3.24)$$

with $x = KU^+$ and $K = 0{\cdot}4$. This formula is supposed to apply for all values of Pr.

In addition to the investigations identified in Table 7, Mizushina *et al.* [23] and Wassel and Catton [24] give special attention to the region adjacent to a fixed wall.

3.4. *Empirical formulae: Table 8*

We shall not consider the many investigations in which Pr_t or Sc_t has been assigned a value independent of the molecular diffusivities and of position within a flow, a procedure most often applied when $Pr, Sc \sim 1$ and the variation in Pr_t, Sc_t is accordingly not large.

Some workers, though aware of attempts to predict the dependence of the eddy-diffusivity ratio on molecular diffusivities and position within the flow, have rejected these analytical results, in view of the contradictions among the several models, between models and

Table 8. Empirical formulae

Authors	Functional form	Validity
Rotta [46]	$f(y/\delta)$	$Pr \sim 1$
Kunz and Yerazunis [55]	$f(\varepsilon_m/\kappa)$	$Pr \ll 1$
Gräber [56]	$f(Pr)$	$1000 > Pr > 0{\cdot}7$
Quarmby and Quirk [9]	$f(y/R)$	$Pr > 0{\cdot}7$

experiment, and between different experiments. Thus Kunz and Yerazunis [55] adopted

$$Pr_t = (2/3)\exp\{0{\cdot}90/(\varepsilon_m/\kappa)^{0{\cdot}64}\} \tag{3.25}$$

for pipe flow, accepting deviations $\pm 0{\cdot}5$ from the experimental values. Again, in view of the scatter in their measurements, Quarmby and Quirk [9, 10] felt that it was impossible to isolate the dependence of Pr_t, Sc_t on the molecular values and Reynolds number. However, they did find an important variation across the pipe, and represented it by

$$Pr_t = (1+400^{-y/R})^{-1}. \tag{3.26}$$

This gives $Pr_t \simeq \frac{1}{2}$ near the wall and $Pr_t \simeq 1$ in the core; the scatter around this result is $\pm 0{\cdot}1$ in the core and rather more in the wall layer.

Rotta [46] considered the consequences of adopting a number of alternative variations of Pr_t through boundary layers in air, including an expression that he found to represent the experiments of Ludwieg and of Johnson:

$$Pr_t = 0{\cdot}9 - 0{\cdot}4(y/\delta)^2 \tag{3.27}$$

(Here δ is the overall boundary-layer thickness.) In fact, the result (3.27) is usually quoted with the constants given the values 0·95 and 0·45. The trend suggested is opposite to that of equation (3.26), the near-wall value being close to unity, while $Pr_t \simeq \frac{1}{2}$ near the edge. Looking at some of the models introduced earlier, we note that they too are in conflict regarding the changes in Pr_t as the wall is approached; compare, for example, equations (2.11), (2.13), (3.23) and (3.24). This confusion reflects the contradictory nature of the experimental evidence.

The last few results have purported to show how Pr_t varies within a particular flow, but did not allow for any dependence on the molecular Prandtl number. At the other extreme, Gräber [56] took no account of position within the flow, but did allow for dependence on molecular diffusivities, through the formula

$$Pr_t^{-1} = 0{\cdot}91 + 0{\cdot}13Pr^{0{\cdot}545} \tag{3.28}$$

applicable for $0{\cdot}7 < Pr < 100$.

4. ASSESSMENT

We now face the task of weighing up the advantages and disadvantages of the results introduced above. The assessment can be based on either practical or fundamental considerations. The former are summed up in the questions: How successful is the model in describing experimental data or situations of practical interest? Is the result adaptable to varied flows, for instance, the wall layer, free turbulence, or axisymmetric flow? How easily can the result be modified when further data become available? How easily can it be derived and used? The first of these questions is the most important, but the variability of the evidence currently available prevents us from using it as anything more than a very coarse filter. This is particularly true of results for the wall region.

In making an assessment from the fundamental point of view we must bear in mind such questions as: What does the model tell us about turbulent transport processes? Does the development suggest fundamental experiments that might clarify the nature of these processes? Does the development make use of measurable quantities? Are the results for limiting cases (such as $Pr \to 0$, 1 and ∞, $y \to 0$ and $Re \to \infty$) sensible? Here too the first question is perhaps the most important. But most of these investigations tell us nothing new about the small-scale transfer processes. Even the more modest final question provides little guidance, for we are by no means sure what the limiting behaviour should be.

In view of the rather cursory discussion that has been possible in this paper, the following remarks are advanced tentatively, but a somewhat more detailed consideration has been given elsewhere [57].

4.1. *Mixing-length models*

From the fundamental point of view, these are not successful: as indicated earlier, they tell us little about the primary transfer processes within the turbulence, while the model of mixing is so crude that it cannot be related directly to correlation and spectrum measurements.

For practical purposes, we must favour those results that contain a number of adjustable constants, and whose form is generally realistic. On this basis it is suggested that the models of Mizushina *et al.*, Wassel and Catton, Azer and Chao, and Buleev are those most suitable for application in engineering practice. Each has its strengths and failings. For example, Wassel and Catton's result is easy to derive and use, but gives no guide as to the changes from one flow species to the next. On the other hand, Buleev's derivation is necessarily complicated, but does provide, in principle at least, a means of dealing with varied boundary geometries.

4.2. *Alternative approaches*

The purely empirical results of Section 3.4 and Table 8 offer neither practical nor fundamental advantages;

their chief function is to remind us how little confidence can be placed in any limited group of measurements.

The wall-layer models considered in Section 3.3 and Table 7 should, if properly tuned with appropriate constants, give a reasonable picture of the wall region. But there is no substantial evidence that they are superior, even for that region, to the more flexible of the mixing-length models. The renewal-penetration models do have a fundamental advantage, however, in being related to a specific and not unrealistic picture of events in the viscous layer.

The diffusivity models of Section 3.2 and Table 6 have the desirable feature of being able to account for non-diffusive transfers within the turbulence, as in equations (3.16, 18). They are rather easy to derive, provided that one accepts the basic integral for the eddy diffusivity, and can be invested with a number of adjustable parameters. On the whole, this starting point seems to retain the desirable features of the mixing-length approach, while being somewhat closer to reality, in that measurable quantities are considered.

Finally, we turn to the formal analyses of Section 3.1 and Table 4. Potentially, these can account for pressure interactions and for diffusion and convection within the turbulence. At present, they describe a very limited class of motions, but we might hope that these results will indicate the limiting behaviour to be expected in other cases. Dunn and Reid's results (3.4) suggest that

$$\begin{aligned} Pr_t &\to \text{finite constant} \quad \text{for} \quad Pr \gg 1 \\ Pr_t &\propto Pr^{-1} \quad\quad\quad\quad \text{for} \quad Pr \ll 1. \end{aligned} \tag{4.1}$$

Deissler's results, given in Table 5, suggest that

$$Pr_t \to \text{finite constant} \quad \text{for} \quad \varepsilon_m/\nu \gg 1. \tag{4.2}$$

Many of the models considered above do display these characteristics, for example, those of Jenkins, Deissler, Wassel and Catton, Mizushina *et al.*, and Tyldesley. Others display the predicted behaviour when $Pr, \varepsilon_m/\nu \gg 1$, but depart from it in that

$$Pr_t \propto Pr^{-n} \text{ (with } n < 1) \quad \text{for} \quad Pr \ll 1. \tag{4.3}$$

For instance, the models of Aoki, Na and Habib, Reynolds, and Azer and Chao give $n = 0{\cdot}2$, $0{\cdot}5$, $0{\cdot}5$, $0{\cdot}58$, respectively. The results of Thomas, Blom, and Kunz and Yerazunis display even more varied limiting behaviour; such extreme predictions are almost certainly incorrect. However, it is difficult to know how seriously to take minor departures, such as (4.3), from the pattern of the formal results. It is quite possible that maintained turbulence differs fundamentally from the decaying homogeneous turbulence for which the results (4.1, 2) are applicable. Indeed, Deissler's results of Table 5 suggest that this is so.

It need hardly be stressed that the formal analyses are of great fundamental interest. Now we see that they have immediate practical application in providing criteria for assessing the structure of semi-empirical formulae.

5. CONCLUSIONS

There seems to be little point in devising further simplistic models of turbulent transport processes, in particular, those based on the mixing-length concept. A variety of functional forms is available, each with several adjustable constants, and with these it should be possible to represent experimental data with adequate precision. Many of the models display the same limiting behaviour, and this suggests that appropriate choices of constants will produce rather similar variations overall.

To describe the flow near a wall, the form $Pr_t = f(Pr, \varepsilon_m/\nu)$ is almost certainly inadequate. The roles of intensity and of position within the flow must be separately accounted for, as in $Pr_t = f(Pr, \varepsilon_m/\nu, y/y_f)$ or $f(Pr, \varepsilon_m/\nu, y/R)$.

Few attempts have been made to predict the marked differences between free turbulence and wall turbulence, and between plane and round flows, although Buleev and Tyldesley have taken some steps in this direction. At present, it seems to be necessary to devise a special law, or to select a special set of empirical constants, for each flow species. Separate treatment will be required for highly intermittent regions of free-turbulent flows and boundary layers, but no useful predictions are currently available.

The vital problem of predicting the effect of pressure interactions on momentum transport has been avoided, save in those analyses that follow directly from Reynolds equations, and apply in the main to decaying turbulence. This is also true of the more general problem of accounting for turbulent mixing that is not adequately represented by gradient diffusion. Less restricted results from formal analysis would be very welcome, but it is not easy to be optimistic in view of the practical difficulties of computation and the fundamental problem of achieving closure.

While it seems unlikely that the purely analytical and purely empirical approaches will make rapid progress, there are intermediate positions which offer more hope. The renewal-penetration models for the wall layer (Section 3.3) and the more widely applicable diffusivity models (Section 3.2) are based on fairly specific and realistic pictures of turbulent activity. Hence intermediate stages of these analyses can be tested against experiment, and progressive improvement is possible.

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LA PREVISION DES NOMBRES DE PRANDTL ET DE SCHMIDT TURBULENTS

Résumé—L'article étudie plus de trente façons de prévoir la relation entre les transferts turbulents de quantité de mouvement et d'un contaminant passif tel que la chaleur ou une substance dissoute. Les modèles sont répartis en sept classes suivant la méthode d'obtention et le champ d'application. Trois des classes comprennent des modifications du modèle le plus simple de longueur de mélange afin de tenir compte de la diffusion lors du mouvement latéral de l'élément fluide qui est sensé transporter les entités transférées. Les quatre autres classes sont plus hétérogènes: analyses formelles basées sur les équations de Reynolds; résultats déduits d'expressions variées des diffusivités turbulentes; plusieurs genres de modèles applicables en particulier aux écoulements pariétaux; et des formules purement empiriques représentant des données limitées.

Une tentative a été faite afin d'évaluer l'utilité et les possibilités du développement de ces modèles, à la fois d'un point de vue pratique, celui d'imaginer des formules précises de transfert thermique et massique, et d'un point de vue fondamental, qui est de parvenir à la compréhension des mécanismes réels du transfert. Il y a une pléthore de formules qui peuvent, moyennant un choix convenable des constantes empiriques, représenter les caractères essentiels des données expérimentales. Cependant, seules les analyses formelles rendent compte de manière cohérente des intéractions de pression qui influencent le transfert de quantité de mouvement. Actuellement, les résultats formels ne sont valables que dans le cas dégénéré d'une faible turbulence en décroissance, bien qu'ils s'avèrent effectivement utiles pour suggérer le comportement limite des écoulements plus généraux.

DIE VORAUSSAGE DER TURBULENTEN PRANDTL- UND SCHMIDT-ZAHLEN

Zusammenfassung—Zur Berechnung der Beziehungen zwischen dem turbulenten Impuls-, Wärme- und Stoffaustausch werden in dieser Arbeit mehr als dreißig Möglichkeiten angegeben. Je nach Art der Herleitung oder des Anwendungsbereichs werden die Modelle in sieben Klassen unterteilt. Das einfachste Mischungslängenmodell ist in drei Klassen enthalten; es berücksichtigt die Diffusion während der Querbewegung des Fluidelements. Die anderen vier Klassen sind unterschiedlicher: auf den Gleichungen von Reynolds basierende formale Berechnungen, Ergebnisse, die von verschiedenen Ausdrücken für die Diffusion in Wirbeln abgeleitet wurden, einige Modelle, die besonders auf Grenzschichten an Wänden anwendbar sind und nur in begrenzten Bereichen gültige empirische Formeln.

Es wurde ein Versuch unternommen, die Nützlichkeit und Anwendungsmöglichkeikt dieser Modelle zu bewerten, zum einen vom praktischen Standpunkt aus, um genaue Wärme- und Stofftransportgleichungen zu erhalten und zum anderen vom grundsätzlichen Standpunkt aus, um den wirklichen Übertragungsmechanismus kennenzulernen. Durch passende Wahl von empirischen Konstanten ist es mit einer großen Anzahl von Gleichungen möglich, die Haupteigenschaften von experimentellen Daten darzustellen. Nur die formalen Ansätze berücksichtigen in übereinstimmender Weise die Wechselwirkungen des Druckes, welche den Impulstransport beeinflussen. Gegenwärtig sind Ergebnisse nur verfügbar für den entarteten Fall der schwachen, abnehmenden Turbulenz; gleichwohl erweisen sich die Ergebnisse als hilfreich, um das Verhalten in allgemeineren Strömungen näherungsweise anzugeben.

РАСЧЕТ ТУРБУЛЕНТНЫХ ЧИСЕЛ ПРАНДТЛЯ И ШМИДТА

Аннотация — В статье анализируются более тридцати методов расчета зависимости между турбулентным переносом импульса и переносом пассивной примеси типа температуры или концентрации вещества. Методы разделены на семь классов в зависимости от способа получения коэффициентов переноса области применения результатов расчета. Три класса содержат модификации самой простой модели длины пути смешения для учета процесса турбулентной диффузии при поперечном движении элемента жидкости, являющегося носителем транспортабельной субстанции. Остальные четыре класса являются более разнородными: они включают исследования, базирующиеся на формальном анализе переноса на основании уравнений Рейнольдса, применении различных выражений для турбулентной температуропроводности, использовании различных моделей «законов» в пристенной области, а также применении чисто эмпирических формул, обобщающих ограниченные экспериментальные данные.

Сделана попытка оценить необходимость перспективности дальнейшей разработки этих моделей как в практических целях, например для разработки некоторых адекватных соотношений теории тепло- и массообмена, так и фундаментальных, например для выяснения механизмов переноса. Имеется множество формул, с помощью которых, при соответствующем подборе эмпирических констант, можно приближенно описывать экспериментальные данные. Однако, только формальный качественный анализ позволяет учесть взаимодействие поля давления и скорости, влияющее на перенос импульса. В настоящее время имеется анализ процессов переноса только для случая слабой вырождающейся турбулентности, хотя он имеет отношение скорее к предельному случаю существования турбулентности вообще, чем к реальным турбулентным потокам.